MRI-PHY/97-01, hep-th/9701132

Solitons in 1+1 Dimensional Gauged Sigma Models

Pijush K. Ghosh*

The Mehta Research Institute of Mathematics & Mathematical Physics,

Chhatnag Road, Jhusi, Allahabad-221506, U.P., INDIA.

Abstract

We study soliton solutions in 1+1 dimensional gauged sigma models, obtained by dimensional reduction from its 2+1 dimensional counterparts. We show that the Bogomol'nyi bound of these models can be expressed in terms of two conserved charges in a similar way to that of the BPS dyons in $3+1$ dimensions. Purely magnetic vortices of the 2+1 dimensional completely gauged sigma model appear as charged solitons in the corresponding 1+1 dimensional theory. The scale invariance of these solitons is also broken because of the dimensional reduction. We obtain exact static soliton solutions of these models saturating the Bogomol'nyi bound.

PACS number(s): 11.15.-q; 11.10.Kk; 11.10.Lm, 03.65.Ge

Typeset using REVTEX

* E-mail: pijush@mri.ernet.in

I. INTRODUCTION

Recently, there has been much interest in the study of soliton solutions in 2+1 dimensional gauged sigma models [1–7]. These models can be viewed as a low energy effective action of certain gauged linear sigma models with judiciously chosen Higgs potential. The self-dual soliton solutions of the completely gauged sigma model with pure Chern-Simons(CS) dynamics are scale invariant [1]. In fact, the scalar multiplet is exactly equivalent to that of the usual sigma model [8]. Moreover, purely magnetic vortices can be obtained from this completely gauged sigma model after a suitable reduction of the non-Abelian gauge group, as in the case of 't Hooft-Polyakov monopole [9]. Unlike in the monopole case, these magnetic vortices can also be obtained from an Abelian theory, since the Bianchi identity for the $U(1)$ invariant field strength is satisfied in $2+1$ dimensions. All these results can be generalized to the completely gauged CP^N models, either with pure CS [1] or Yang-Mills CS dynamics [2].

Solitons of the 2+1 dimensional $U(1)$ gauged sigma models are not scale invariant, because of the presence of a $U(1)$ invariant potential term [3,5]. No exact solution is known for any of these models. However, it is known that these models admit a variety of new soliton solutions [3,5,7]. One such peculiar feature is that the magnetic flux of the topological solitons in the symmetric phase of the theory is not necessarily quantized like vortices in the Abelian Higgs models. The quantization of the magnetic flux is recovered in the asymmetric phase. These sigma models with pure CS dynamics admit both topological as well as nontopological soliton solutions, as in the case of self-dual $U(1)$ CS Higgs theory. In fact, these $U(1)$ gauged sigma models can be reduced to the self-dual $U(1)$ CS Higgs theory [10] in certain limits. All these results have been generalized to the gauged CP^N case with pure CS dynamics [6], where a proper subgroup

of the global $SU(N+1)$ is completely gauged.

The 1+1 dimensional sigma models have many properties in common with the 3+1 dimensional Yang-Mills-Higgs(YMH) theory. One such remarkable feature is that a class of $1+1$ dimensional sigma models admit Q-kinks [11,12] with similar properties to those of the BPS dyons of YMH theory [13]. These 1+1 dimensional sigma models are obtained by dimensional reduction from their 2+1 dimensional counterparts. It is reasonable at this point to study dimensionally reduced version of the $2+1$ dimensional gauged sigma models.

The purpose of this paper is to study soliton solutions in 1+1 dimensional gauged sigma models obtained by dimensional reduction from 2+1 dimensions. In particular, we consider dimensionally reduced version of two different 2+1 dimensional models with pure CS dynamics, (i) the completely gauged sigma model [1] and (ii) the $U(1)$ gauged sigma model [5]. We find that the Bogomol'nyi bound [14] for both of these models are of BPS type, namely, the lower bound on the energy is expressed as a linear combination of the topological charge and the Noether charge. Moreover, the scale invariant soliton solutions, describing purely magnetic vortices, of the 2+1 dimensional completely gauged sigma model, appear as solitons solutions with nonzero Noether charge and a definite scale, in the corresponding 1+1 dimensional theory. This resembles the way BPS dyons in 3+1 dimensions can be obtained from four dimensional Eucledian self-dual Yang-Mills theory. Such an analog already exists in the framework of certain 1+1 dimensional sigma models admitting Q-kinks [11] which are necessarily time dependent solutions. However, in our case the similarity is between static solitons of the completely gauged model and 3+1 dimensional dyons. Recently, it has been shown that the dimensionally reduced version of self-dual $U(1)$ CS Higgs theory also shares similar properties [15]. Unfortunately, the soliton solutions of 2+1 dimensional $U(1)$ CS Higgs theory are not

scale invariant. Thus, the completely gauged sigma model studied in this paper have more similarities with BPS dyons than any other existing models. We find all the static, exact soliton solutions of the completely gauged sigma model saturating the Bogomol'nyi bound. We are able to obtain only a class of exact soliton solutions for the $U(1)$ gauged sigma model. The soliton solutions of both of these models are domain walls, interpolating between different symmetric and asymmetric vacua.

The plan of this paper is the following. First, we introduce and study the 1+1 dimensional completely gauged sigma model in Sec. II. In particular, we describe the dimensional reduction procedure and obtain the 1+1 dimensional model from the 2+1 dimensional completely gauged sigma model. We obtain the Bogomol'nyi bound of this model and present all static, exact solutions of the Bogomol'nyi equations. In Sec. III, we discuss the $U(1)$ gauged sigma model and obtain a set of exact, static soliton solutions saturating the Bogomol'nyi bound. Finally, we give a summary of the results obtained in this paper and discuss on possible directions to be explored in Sec. IV. In appendix A, we present some more exact soliton solutions saturating the Bogomol'nyi bound for the $U(1)$ gauged sigma model.

II. $O(3)$ GAUGED SIGMA MODEL

The self-dual completely gauged sigma model in 2+1 dimensions is given by [1],

$$\mathcal{L}_0 = \frac{1}{2} D_\mu \chi^a D^\mu \chi^a + \frac{\kappa}{4} \epsilon^{\mu\nu\lambda} \left(F^a_{\mu\nu} A^a_\lambda - \frac{e}{3} \epsilon^{abc} A^a_\mu A^b_\nu A^c_\lambda \right). \tag{1}$$

The real scalar field χ has three components and is constrained to lie on a unit sphere in the internal space, i.e., $\chi^a \chi^a = 1$. The covariant derivative and the field strengths F^a_{01} are defined as,

$$D_\mu \chi^a = \partial_\mu \chi^a + e \epsilon^{abc} A^b_\mu \chi^c, \quad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + e \epsilon^{abc} A^b_\mu A^c_\nu. \tag{2}$$

The self-dual equations of (1) are,

$$D_0\chi^a = 0, \quad D_i\chi^a \pm \epsilon_{ij}\epsilon^{abc}\chi^b D_j\chi^c = 0. \tag{3}$$

The self-dual soliton solutions of (1) are characterized by the energy $E = 4\pi q_0$, where the topological charge q_0 is defined as,

$$q_0 = \int d^2x k_0, \quad k_\mu = \frac{1}{2}\epsilon_{\mu\nu\lambda}\mathcal{F}^{\nu\lambda}, \quad \mathcal{F}_{\mu\nu} = \epsilon^{abc}D_\mu\chi^a D_\nu\chi^b\chi^c - eF^a_{\mu\nu}\chi^a. \tag{4}$$

Here, k_μ is the gauge invariant topological current and $\mathcal{F}_{\mu\nu}$ is the $U(1)$ invariant field strength. The conservation of the topological current automatically implies the Bianchi identity for $\mathcal{F}_{\mu\nu}$. Note that the electric field $\mathcal{F}_{01}$ vanishes for the self-dual field configurations, since the Gauss law implies $F^a_{\mu\nu}\chi^a = 0$. However, the magnetic field is nonzero and it describes Liouville vortex. These self-dual field configurations are of zero Noether charge. As a consequence, the spatial components of the gauge fields are pure gauges and the second equation of (4) can be exactly mapped into the corresponding equation of the usual sigma model. Thus, the solitons are scale invariant.

We now dimensionally reduce the model (1) to 1+1 dimensions. We take all the field variables to be independent of the second coordinate and identify the second component of the gauge field with a triplet M^a, $A^a_2 = M^a$. Following the standard procedure, we have the $1+1$ dimensional completely gauged sigma model,

$$\mathcal{L}_1 = \frac{1}{2}(D_\mu\chi^a)^2 - e^2\left(M^aM^a - (M^a\chi^a)^2\right) + \kappa M^aF^a_{01}, \quad \mu = 0, 1. \tag{5}$$

Note the appearance of an interaction term. The vacuum is characterized by those field configurations, for which the norm of the triplet M is exactly equal to the square of its own projection along χ.

The equations of motion of (5) are,

$$\kappa F^a_{01} = e^2\left[M^a - \chi^a(M^b\chi^b)\right], \quad \kappa D_0 M^a = K^a_1, \quad \kappa D_1 M^a = K^a_0,$$

$$D_\mu K^{a,\mu} = e^3\epsilon^{abc}M^b\chi^c(M^d\chi^d), \quad K^a_\mu = e\epsilon^{abc}D_\mu\chi^b\ \chi^c, \tag{6}$$

where a prime denotes differentiation with respect to the space coordinate. Note that $F^a_{01}\chi^a = 0$. Also, $\chi^a D_\mu M^a = 0$, since $K^a_\mu\chi^a = 0$.

We now define two conserved charges Y_1 and Z_1,

$$Y_1 = \kappa\int dx\,[M^a M^a]', \quad Z_1 = 2e\int dx\,[\chi^a M^a]'. \tag{7}$$

These two charges, Y_1 and Z_1, can be identified as topological and Noether charges respectively. The reason for such an identification is the following. The momentum along the compactified direction, P_y, is a conserved quantity in the $2+1$ dimensional theory, because of the translational invariance. This quantity remains conserved in the corresponding $1+1$ dimensional model also. Using the Gauss law, we find,

$$P_y = -\frac{\kappa}{2}\int dx\partial_1\left(M^a M^a\right). \tag{8}$$

Thus, Y_1 can be identified as the topological charge. In order to identify Z_1 as the Noether charge, recall that we are dealing with pure CS dynamics. Consequently, the topological current k_μ , as defined in the second equation of (4), can be identified as the $U(1)$ current in the asymmetric phase of the $2+1$ dimensional theory. This $U(1)$ charge remains conserved in the corresponding $1+1$ dimensional theory also. In particular, the topological charge k_0 (or equivalently the $U(1)$ Noether charge) reduces to the following expression after the dimensional reduction,

$$\mathcal{F}_{12} = -eD_1\chi^a M^a = -e\partial_1\left(\chi^a M^a\right). \tag{9}$$

Thus, Z_1 is identified as the Noether charge.

The energy functional can be written as,

$$E = \int dx \left[\left(D_0 \chi^a \pm e \cos\beta \epsilon^{abc} M^b \chi^c \right)^2 + \left(D_1 \chi^a \mp e \sin\beta \left\{ M^a - \chi^a \left(\chi^b M^b \right) \right\} \right)^2 \right]$$
$$\pm (Y_1 \cos\beta + Z_1 \sin\beta). \tag{10}$$

The lower bound on the energy functional is saturated for the solutions of the following first order equations,

$$D_0 \chi^a \pm e \cos\beta \epsilon^{abc} M^b \chi^c = 0, \quad D_1 \chi^a \mp e \sin\beta \left[M^a - \chi^a \left(\chi^b M^b \right) \right] = 0. \tag{11}$$

These two first order equations are consistent with the second order field equations. The gauge potential A_0^a is determined in terms of χ^a and M^a from the first equation of (6) and (11) respectively as,

$$A_0^a = \mp \frac{e}{\kappa} \sin\beta \chi^a \mp \cos\beta M^a. \tag{12}$$

Using the Gauss law and the first Bogomol'nyi equation of (11), we have,

$$D_1 M^a \mp \frac{e^2}{\kappa} \cos\beta \left[M^a - \chi^a (\chi^b M^b) \right] = 0. \tag{13}$$

Multiplying the second equation of (11) and Eq. (13), respectively by M^a, we find,

$$\partial_1 (M^a M^a) = \pm 2 \cos\beta \left[M^a M^a - (M^a \chi^a)^2 \right],$$
$$\partial_1 (M^a \chi^a) = \pm \sin\beta \left[M^a M^a - (M^a \chi^a)^2 \right]. \tag{14}$$

We have made the following rescaling of the field variables,

$$M^a \rightarrow \frac{e}{\kappa} M^a, \quad A_1^a \rightarrow \frac{e}{\kappa} A_1^a, x \rightarrow \frac{\kappa}{e^2}, \tag{15}$$

while deriving the equation (14). Note that all the field variables as well as the space coordinate are now dimensionless quantity.

We now discuss the solutions of (14) for $\beta \neq \frac{m\pi}{2}$ and $\beta = \frac{m\pi}{2}$ separately, where m is any integer.

(a) $\beta \neq \frac{m\pi}{2}$: Note that,

$$M^a \chi^a = \frac{1}{2} \tan\beta \, (M^a M^a - b) \tag{16}$$

for $\beta \neq \frac{m\pi}{2}$, where b is the integration constant. Using the relation (16) in the first equation of (14), $M^a M^a$ can be determined completely,

$$M^a M^a = \pm 2\delta \cot^2\beta \tanh\left[\delta \cos\beta (x - x_0)\right] + b + 2\cot^2\beta, \quad \delta = (1 + b \tan^2\beta)^{\frac{1}{2}}, \tag{17}$$

where x_0 is an integration constant. The finite energy field configurations demand that the integration constant $b \geq -\cot^2\beta$. The asymptotic values of $M^a M^a$ and $\chi^a M^a$ are,

$$\begin{aligned} (M^a M^a)_+ &= 2(1 \pm \delta)\cot^2\beta + b, \quad (M^a M^a)_- = 2(1 \mp \delta)\cot^2\beta + b, \\ (\chi^a M^a)_+ &= (1 \pm \delta)\cot\beta, \quad (\chi^a M^a)_- = (1 \mp \delta)\cot\beta, \end{aligned} \tag{18}$$

where a subscript 'plus' or 'minus' denotes the value of the quantity inside the bracket at $x = \infty$ or $x = -\infty$, respectively. We will follow this notation throughout this paper. The topological charge, the Noether charge and the energy are,

$$Y_1 = \pm \frac{4e^2}{\kappa} \delta \cot^2\beta, \quad Z_1 = \pm 4\frac{e^2}{\kappa} \delta \cot\beta, \quad E = \frac{4e^2}{\kappa} \delta \frac{\cot\beta}{\sin\beta}. \tag{19}$$

Note that $Y_1 = E\cos\beta$ and $Z_1 = E\sin\beta$. As a result, $E = \sqrt{Y_1^2 + Z_1^2}$, very much like dyons in YMH theory. Also, note that Y_1, Z_1 and E are dependent on the integration constant b through δ. For any value of β, one can make all these quantity to be zero by fixing b to take its minimum allowed value, i. e., $b = -\cot^2\beta$, or equivalently $\delta = 0$. This describes trivial vacuum solution. Finite energy nontrivial soliton solutions exist for $\delta > 0$.

We now choose consistently A_1^a to be zero. This implies that χ^a and M^a are not independent,

$$M^a = \eta_0^a + \cot\beta \chi^a, \tag{20}$$

where η_0^a are three independent constants. Plugging back this expression into the second Bogomol'nyi equation and making use of Eq. (17), we determine χ^a,

$$\chi^a = \pm \frac{\eta_0^a}{\delta} \tan\beta \tanh\left[\delta\cos\beta(x - x_0)\right] \pm \theta^a \mathrm{sech}[\delta\cos\beta(x - x_0)]. \tag{21}$$

The constants η_0^a and θ^a satisfy the following relations in order to maintain the unit norm of χ^a,

$$\theta^a\theta^a = 1, \quad \theta^a\eta_0^a = 0, \quad \eta_0^a\eta_0^a = \delta^2\cot^2\beta. \tag{22}$$

One particular choice of θ and η_0 satisfying the relation (22) is $\theta = (0,0,1)$ and $\eta_0 = (\delta\cos\psi\cot\beta, \delta\sin\psi\cot\beta, 0)$, where ψ is an arbitrary angle. In general, θ^a can be parameterized as the coordinates of unit sphere and η_0^a as the coordinates of a sphere of radius $\delta\cot\beta$. This reduces the number of constants to four. This can further be reduced to three by imposing the condition $\theta^a\eta_0^a = 0$.

(b) $\beta = \frac{m\pi}{2}$: We now discuss the special case $\beta = \frac{(2m+1)\pi}{2}$. All the components of M^a are constant, $M^a = \xi^a$. The quantity $\xi^a\chi^a$ is determined as,

$$M^a\chi^a = \xi^a\chi^a = \mp p\tanh\left[(x - x_0)p\right], \quad p = \xi^a\xi^a. \tag{23}$$

Using this expression, we find,

$$\chi^a = \mp\left[\frac{\xi^a}{p}\tanh\left[(x - x_0)p\right] + \eta^a\mathrm{sech}[(x - x_0)p]\right], \tag{24}$$

where η^a are three different integration constants having the properties,

$$\eta^a\xi^a = 0, \quad \eta^a\eta^a = 1. \tag{25}$$

These properties of η^a and ξ^a are necessary in order to maintain the unit norm of χ^a. Now note that $M^a\chi^a$ goes to $\mp p$ at $x = +\infty$, while it is ± 1 at $x = -\infty$. Thus, $Z_1 = \mp 2pe$ and $Y_1 = 0$. This is the zero topological charge sector. The purely topological sector is given by $\beta = m\pi$. In this case, the role of χ^a gets exchanged with M^a.

III. $U(1)$ GAUGED SIGMA MODEL

The self-dual $U(1)$ gauged sigma model with pure CS dynamics in 2+1 dimensions is given by [5],

$$\mathcal{L}_3 = \frac{1}{2} D_\mu \vec{\phi} . D^\mu \vec{\phi} + \frac{k}{4} \epsilon^{\mu\nu\lambda} A_\mu F_{\nu\lambda} - \frac{1}{2k^2} \left(\phi_1^2 + \phi_2^2 \right) (v - \phi_3)^2 , \tag{26}$$

where $\vec{\phi}$ is a three component real scalar field, $\vec{\phi} = \hat{n}_1 \phi_1 + \hat{n}_2 \phi_2 + \hat{n}_3 \phi_3$, with unit norm in the internal space. The covariant derivative is defined as,

$$D_\mu \vec{\phi} = \partial_\mu \vec{\phi} + A_\mu \hat{n}_3 \times \vec{\phi}, \tag{27}$$

and the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The scalar potential in (26) has three degenerate minima for $0 \leq | v | < 1$. The symmetric phases are described by $\phi_3 = \pm 1$, while the asymmetric phase is at $\phi_3 = v$. These three minima merge into two for $| v | \geq 1$ and we are left with only symmetric phases of the theory. The Lagrangian (26) admits topological as well nontopological soliton solutions with nonzero Noether charge.

We now obtain the 1+1 dimensional model corresponding to (26) , following the same procedure as in the case of the completely gauged sigma model, with the identification $A_2 = N$,

$$\mathcal{L}_4 = \frac{1}{2} D_\mu \vec{\phi} . D^\mu \vec{\phi} + \kappa N F_{01} - \frac{1}{2} N^2 (\phi_1^2 + \phi_2^2) - \frac{1}{2\kappa^2} (\phi_1^2 + \phi_2^2)(v - \phi_3)^2, \quad \mu = 0, 1. \tag{28}$$

The potential in (28) has three degenerate minima, (i) $\phi_3 = \pm 1$, $N \equiv$ arbitrary constant and (ii) $\phi_3 = v, N = 0$ for $0 \leq | v | < 1$. For $| v | \geq 1$, there are only two degenerate minima in the symmetric phase.

The equations of motion which follow from Eq. (28) are,

$$\kappa F_{01} = N(\phi_1^2 + \phi_2^2), \quad \kappa N' = j_0, \quad \kappa \partial_0 N = j_1, \quad j_\mu = \phi_2 D_\mu \phi_1 - \phi_1 D_\mu \phi_2, \tag{29}$$

$$D_\mu \vec{J}^\mu = -\left(\hat{n}_3 \times \vec{\phi}\right)\left(N^2\phi_3 + \frac{1}{k^2}(v-\phi_3)(v\phi_3 + 1 - 2\phi_3^2)\right). \qquad (30)$$

The current $\vec{J}_\mu = \vec{\phi} \times D_\mu \vec{\phi}$ and the $U(1)$ current is given by $j_\mu = -\vec{J}_\mu.\hat{n}_3$. The Noether charge q is determined in terms of the asymptotic behaviour of N from (29) as,

$$q = \kappa \int dx N' = \kappa[N_+ - N_-]. \qquad (31)$$

Note that the nonzero Noether charge sectors are characterized by, $N_+ \neq N_-$.

We define two different conserved charges as follows,

$$Y_2 = \frac{1}{2\kappa}\int dx \left[(v-\phi_3)^2 - \kappa^2 N^2\right]', \quad Z_2 = \int dx \left[N(v-\phi_3)\right]'. \qquad (32)$$

We identify Y_2 and Z_2 as topological and Noether charge , respectively. To see this, note that the momentum along the compactified dimension, $P_y = \int dx D_0\vec{\phi}.D_2\vec{\phi}$, can be expressed in terms of the asymptotic values of N with the help of Gauss law. In particular, $P_y = \frac{q}{2}(N_+ + N_-) = q\bar{N}$. We rewrite Y_2 in terms of the asymptotic values of the field variables,

$$Y_2 = \frac{1}{2\kappa}\left[(v-\phi_3)_+^2 - (v-\phi_3)_-^2 - 2\kappa P_y\right]. \qquad (33)$$

Notice that $Z_2 = \frac{v\mp 1}{\kappa}q$, in case, ϕ_3 interpolates from any one of the symmetric vacua to the asymmetric vacuum. This is also true when ϕ_3 interpolates between the same symmetric phase. However, Z_2 receives an extra contribution, $Z_2 = \frac{v}{\kappa}q \mp 2\frac{P_y}{q}$, in case it interpolates between the different symmetric phases. In a broad sense, Z_2 can thus be regarded as the Noether charge.

The energy functional corresponding to (28) is,

$$E = \frac{1}{2}\int dx \left[D_0\vec{\phi}.D_0\vec{\phi} + D_1\vec{\phi}.D_1\vec{\phi} + N^2(\phi_1^2+\phi_2^2) + \frac{1}{\kappa^2}(\phi_1^2+\phi_2^2)(v-\phi_3)^2\right]. \qquad (34)$$

The term $\kappa_0 N F_{01}$ do not contribute to the energy functional, since it is first order in space-time derivative. Let us now introduce two orthogonal vectors $\vec{A}$ and $\vec{B}$ as follows,

$$\vec{B} = \hat{n}_3 \times \vec{\phi}, \quad \vec{A} = \vec{\phi} \times \vec{B}. \tag{35}$$

These two vectors have the following properties,

$$\vec{A}.\vec{A} = \vec{B}.\vec{B} = \phi_1^2 + \phi_2^2, \quad \vec{A}.\vec{B} = 0, \quad D_\mu\vec{\phi}.\vec{A} = \partial_\mu\phi_3, \quad D_\mu\vec{\phi}.\vec{B} = -j_\mu. \tag{36}$$

Using Eqs. (35) and Eqs. (36), the energy functional (34) can be conveniently rewritten as,

$$E = \frac{1}{2}\int dx \left[\left(D_0\vec{\phi} \mp \vec{B}P\right)^2 + \left(D_1\vec{\phi} \pm \vec{A}Q\right)^2\right] \pm (Y_2 \cos\alpha + Z_2 \sin\alpha), \tag{37}$$

where are P and Q are defined as,

$$P = N\cos\alpha - \frac{1}{\kappa}(v - \phi_3)\sin\alpha, \quad Q = N\sin\alpha + \frac{1}{\kappa}(v - \phi_3)\cos\alpha. \tag{38}$$

Note that the lower bound on the energy functional (37), i.e. , $E \geq \mid Y\cos\alpha + Z\cos\alpha \mid$, is expressed as a linear combination of the Noether charge and the topological charge. This is reminiscent of what happens in the case of dyons in the 3+1 dimensional YMH theory.

The Bogomol'nyi bound is saturated, when the following first order equations hold true,

$$D_0\vec{\phi} \mp \vec{B}P = 0, \quad D_1\vec{\phi} \pm \vec{A}Q = 0. \tag{39}$$

These two first order equations are of course consistent with the field equations (29) and (30). With the help of the first equation of (39), the Gauss law can be conveniently rewritten as,

$$\kappa N' = \mp(\phi_1^2 + \phi_2^2)P. \tag{40}$$

Using the stereographic projection,

$$u_1 = \frac{\phi_1}{1+\phi_3}, \quad u_2 = \frac{\phi_2}{1+\phi_3}, \quad u = u_1 + iu_2, \tag{41}$$

the second equation of (39) is transformed as,

$$(\partial_1 + A_1)u \mp Qu = 0. \tag{42}$$

The gauge potential A_1 is determined in terms of the argument of u, $A_1 = -[Arg(u)]'$ and, hence can be consistently chosen as zero. We get the decoupled second order equation in terms of $\rho = \mid u \mid^2$ after combining Eq. (40) with (42),

$$\frac{\partial^2}{\partial x^2} ln\rho = \frac{\rho}{(1+\rho)^3}\left[(v-1) + (v+1)\rho\right]. \tag{43}$$

We have scaled x as $x \to \frac{\kappa}{\sqrt{8}}x$ in the above equation. Eq. (43) is precisely the one dimensional version of the decoupled equation obtained in the 2+1 dimensional $U(1)$ self-dual gauged sigma model (26). However, no exact solutions is known in the 2+1 dimensional case. Eq. (43) can be written as a first order nonlinear equation,

$$\frac{\partial \rho}{\partial x} = \pm \frac{\rho}{1+\rho}\left[a + b\rho + a_0\rho^2\right]^{\frac{1}{2}}, \tag{44}$$

where $a = a_0 - 2v$, $b = 2(a_0 - v - 1)$ and a_0 is the integration constant. Note that Eq. (44) with the upper sign can be related to the same equation with the lower sign, by changing $x \to -x$. Hence, we will consider the lower sign only now onwards. Once ρ is known from Eq. (44), N can be determined as,

$$N = \frac{1}{\kappa}\left[\frac{\sqrt{2}}{\sin\alpha}\frac{\rho'}{\rho} - \cot\alpha \frac{(v-1)+(v+1)\rho}{1+\rho}\right]. \tag{45}$$

We present some exact solutions of (44) below.

No exact solution of (44) is known for arbitrary a_0 and v. We first try the simplest case $\delta_0 = 4aa_0 - b^2 = 0$. The constants are determined as, $a_0 = \frac{1}{2}(1+v)^2$, $a = \frac{1}{2}(1-v)^2$ and $b = v^2 - 1$. The problem now is to find the solution of the algebraic equation,

$$\rho \left(1 + \frac{b}{2a}\rho\right)^{\gamma-1} = \frac{2a}{b^\gamma} e^{-\sqrt{a}(x-x_0)}, \tag{46}$$

for $a, a_0 > 0$, where $\gamma = \sqrt{\frac{a}{a_0}}$ and x_0 is the integration constant. The most obvious choice now is to choose $\gamma = 1$. However, the solution $\rho = -e^{-\frac{1}{\sqrt{2}}(x-x_0)}$ is not a physical one, because of the presence of a minus sign in front of it. The next possibility, $\gamma = 0$, is ruled out since 'a' is zero in this case. The algebraic equation (46) certainly can be solved for $\gamma = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{2}{3}, \frac{3}{4}$ and their reciprocals. This is because Eq. (46) is at most quartic in ρ for these values of γ. It may be possible to solve (46) for other values of γ also. However, there is no general procedure for finding roots of fifth or higher order polynomial equations and we do not discuss such cases here. Now note that for a fixed γ, v is completely determined. In fact, for each value of γ, v has two different values,

$$v = -\frac{\gamma - 1}{\gamma + 1}, \quad -\frac{\gamma + 1}{\gamma - 1}. \tag{47}$$

These two values of v are reciprocal to each other. Consequently, for a fixed value of γ, once we know a solution at a particular v, we also know the solution at $\frac{1}{v}$. Also, note that $\gamma \rightarrow \frac{1}{\gamma}$ implies $v \rightarrow -v$.

We now present some exact soliton solutions and their asymptotic behaviour for different values of γ,

(i) For $\gamma = \frac{1}{2}$, we have the following two solutions in terms of the variable $X = \sqrt{a}(x - x_0)$,

$$\rho_{1,2} = a e^{-X} \left[e^{-X} \pm \left(e^{-2X} + 4 \right)^{\frac{1}{2}} \right], \tag{48}$$

where $\rho_1(\rho_2)$ denotes the solution with the upper(lower) sign. The field variable N is not nonsingular all over the real line for ρ_2 and, hence is not a finite energy solutions. On the other hand, ρ_1 vanishes at $x = \infty$ and diverges at $x = -\infty$. This implies that ϕ_3 interpolates from -1 to 1. The asymptotics of N for this solution is,

$$N_+ = \frac{1}{\kappa} \left[-\frac{\sqrt{2a}}{\sin\alpha} - (v-1)\cot\alpha \right], \quad N_- = \frac{1}{\kappa} \left[-\frac{2\sqrt{2a}}{\sin\alpha} - (v+1)\cot\alpha \right]. \tag{49}$$

This solution ρ_2 is valid for both $v = \frac{1}{3}$ and $v = 3$.

(ii) We have the following solutions for $\gamma = 2$,

$$\rho_{1,2} = \frac{a}{b}\left[-1 \pm \left(1 + \frac{4}{b}e^{-X}\right)^{\frac{1}{2}}\right]. \tag{50}$$

This solution is valid for both $v = -\frac{1}{3}$ and $v = -3$. However, b is negative for $v = -\frac{1}{3}$ and $\rho_{1,2}$ becomes imaginary for certain values of x. Thus, solutions with $v = -\frac{1}{3}$ can not be physical. On the other hand, ρ_2 goes to $-\frac{2a}{b}$ as $x \to \infty$. This means ϕ_3 goes to $v = -3$ at spatial infinity. Unfortunately, ϕ_3 can not take this value because of the constraint $\vec{\phi}.\vec{\phi} = 1$. Thus, the only acceptable solution is ρ_1 with $v = -3$, which vanishes at one end and diverges in the other end, implying that ϕ_3 interpolates between the symmetric vacua. The behaviour of N corresponding to this solution is,

$$N_+ = \frac{4}{\kappa}\left[\cot\alpha - \frac{1}{\sin\alpha}\right], \quad N_- = -\frac{1}{\kappa\sin\alpha} + \frac{2}{\kappa}\cot\alpha. \tag{51}$$

(iii) For $\gamma = \frac{3}{2}$ and $v = -\frac{1}{5}$, we determine ρ as,

$$\rho = \frac{1}{2} + \frac{1}{4}B^{\frac{1}{3}} + B^{-\frac{1}{3}}, \quad B = \frac{1}{2}\left[16 + 225e^{-2X} + 15e^{-X}\left(32 + 225e^{-2X}\right)^{\frac{1}{2}}\right]. \tag{52}$$

As $x \to \infty$, $\rho \to \frac{3}{2}$ and ρ diverges as $x \to -\infty$. This is the solution interpolating between the symmetric and the asymmetric vacuum. In particular, ϕ_3 interpolates from -1 at $x = -\infty$ to $v = -\frac{1}{5}$ at $x = \infty$. The behaviour of N corresponding to this solution is,

$$N_+ = 0, \quad N_- = -\frac{4}{5\kappa\sin\alpha} - \frac{4}{5\kappa}\cot\alpha. \tag{53}$$

We find exact solutions for (a) $\gamma = \frac{1}{3}, v = 2$, (b) $\gamma = 3, v = -2$, (c) $\gamma = \frac{2}{3}, v = \frac{1}{5}$, (d) $\gamma = \frac{1}{4}, v = \frac{5}{3}$ and (e) $\gamma = \frac{3}{4}, v = 7$. All of these solutions interpolate from -1 to 1 and we present some of these solutions in Appendix A. The solutions for other values of γ and v are not physical in the sense that either they are of infinite energy or they become imaginary over certain region of space.

Let us now consider the case $\delta_0 \neq 0$. The problem again is to solve an algebraic equation similar to (46), but more complicated. We are able to solve this equation only for $v = 0$. For this choice of v, $a = a_0$ and $b = 2(a_0 - 1)$. We have the following expression for ρ with $a_0 > 0$,

$$\rho_{1,2} = \frac{1}{2a_0}\left[A \pm \left(A^2 - 4a_0^2\right)^{\frac{1}{2}}\right], \quad A = 2a_0 \cosh^2\frac{X}{2} + (2a_0 + 1)\sinh^2\frac{X}{2}. \tag{54}$$

ϕ_3 goes to -1 at both the spatial infinities for the solution (54). However, N is not well behaved all over the space for ρ_2. We discard this solution. The asymptotic behaviour of N corresponding to ρ_1 is given by,

$$N_+ = \frac{1}{\kappa}\left[\frac{\sqrt{2}}{\sin\alpha} - \cot\alpha\right], \quad N_- = \frac{1}{\kappa}\left[-\frac{\sqrt{2}}{\sin\alpha} - \cot\alpha\right]. \tag{55}$$

Note that the energy is expressed in terms of the topological and the Noether charge as, $E = \sqrt{Y_2^2 + Z_2^2}$, for all of these solutions. This is exactly like the energy bound in BPS dyons. These solitons are domain walls in nature, interpolating between different symmetric and asymmetric vacua.

IV. SUMMARY AND DISCUSSIONS

In conclusion, we have studied soliton solutions in certain 1+1 dimensional gauged sigma models. These models are obtained by dimensionally reducing 2+1 dimensional self-dual gauged sigma models with pure CS dynamics. We have found a remarkable similarity between these $1+1$ dimensional models and the $3+1$ dimensional YMH theory. In particular, the Bogomol'nyi bound is expressed in terms of the topological and the Noether charge in a similar way to that of the BPS dyons. Moreover, the scale invariant solitons with vanishing Noether charge, of the 2+1 dimensional completely gauged sigma model, have definite scale and nonzero Noether charge in the corresponding

1+1 dimensional theory. This resembles the way BPS dyons can be obtained from four dimensional Eucledian Yang-Mills theory. Such a similarity between Q-kinks and the BPS dyons already exists. However, Q-kinks are necessarily time dependent solutions, while BPS dyons are static, minimum energy solutions of the YMH theory. In our case, the similarity is between the static solitons of gauged sigma model with BPS dyons. Recently, it has been shown that the static solitons of dimensionally reduced self-dual $U(1)$ CS Higgs theory also share similar properties with the BPS dyons [15]. However, the soliton solutions of the self-dual CS theory in 2+1 dimensions are not scale invariant. Thus, the soliton solutions of completely gauged sigma models studied in this paper have more similarities with the BPS dyons than any other existing models. Finally, we have obtained all static, exact soliton solutions of the completely gauged sigma model saturating the Bogomol'nyi bound. On the other hand, we found only a class of exact, static soliton solutions for the $U(1)$ gauged sigma model. The soliton solutions of both of these models are domain walls in nature interpolating between different symmetric and asymmetric vacua.

The models considered in this paper have no kinetic energy term corresponding to the gauge fields and the gauge field equations appear as constraints. This is because they are dimensionally reduced version of $2+1$ dimensional models with pure CS dynamics. In this regard, one might also consider the dimensionally reduced version of the $2+1$ dimensional completely gauged sigma model with both Yang-Mills as well as CS dynamics [2]. The resulting Lagrangian would have not only the gauge field kinetic energy term, but also a kinetic energy term for the triplet M in terms of its covariant derivative and certain interaction term dictated by the $2+1$ dimensional anomalous magnetic moment interaction term. We expect that all the results obtained in this paper will go through in a straightforward way for this case also.

It is known that gauged sigma models can be viewed as a low energy effective action of certain gauged linear sigma models [3]. These linear models are useful in studying different kinds of $2+1$ dimensional soliton solutions in an unified manner. However, in general, it is difficult to analyze the Bogomol'nyi equations arising out of these linear sigma models in detail. The study of 1+1 dimensional version of these models, which are expected to be the gauged linear sigma models corresponding to the models studied in this paper, may shed some light on the 2+1 dimensional problem. Similar considerations also apply for the gauged CP^N models.

ACKNOWLEDGMENTS

I would like to thank Avinash Khare for a careful reading of the manuscript and valuable comments.

APPENDIX A: SOME MORE EXACT SOLUTIONS

In this appendix, we present some more exact solutions of Eq. (46) for different values of γ and v. For all of these solutions, ϕ_3 interpolate between -1 and $+1$. Also, the energy is expressed in terms of the topological and Noether charge as, $E = (Y_2^2 + Z_2^2)^{\frac{1}{2}}$.

(a) $\gamma = \frac{1}{3}$, $v = 2$:

$$\begin{aligned}
\rho &= e^{-3X} + (\frac{2}{3C})^{\frac{1}{3}} e^{-3X} \left(2 + 3e^{-3X}\right) + (\frac{C}{18})^{\frac{1}{3}},\\
C &= 3e^{-3X}\left[1 + 6e^{-3X} + 6e^{-6X} + \left(1 + \frac{4}{3}e^{-3X}\right)^{\frac{1}{2}}\right].
\end{aligned} \tag{A1}$$

(b) $\gamma = 3$, $v = -2$:

$$\rho = -2 + D^{\frac{1}{3}} + D^{-\frac{1}{3}}, \quad D = \frac{1}{2}\left[2 + 3e^{-X} + \sqrt{3}e^{-\frac{X}{2}}\left(4 + 3e^{-X}\right)^{\frac{1}{2}}\right]. \tag{A2}$$

(c) $\gamma = \frac{2}{3}$, $v = \frac{1}{5}$:

$$\rho = \frac{2}{15}(\frac{2}{E})^{\frac{1}{3}}\left[-4e^{-3X} + 2^{\frac{1}{3}}E^{\frac{2}{3}}\right], \quad E = e^{-3X}\left[15 + \left(225 + 32e^{-3X}\right)^{\frac{1}{2}}\right]. \tag{A3}$$

REFERENCES

[1] G. Nardelli, Phys. Rev. Lett. 73 (1994) 2524; Phys. Rev. D 52 (1995) 5944; Y. M. Cho and K. Kimm, Phys. Rev. D 52 (1996) 7325 .

[2] P. K. Ghosh, Phys. Lett B 381 (1996) 237.

[3] B. J. Schroers, Phys. Lett. B 356 (1995) 291; Nucl. Phys. B 475 (1996) 440.

[4] J. Gladikowski, B. M. A. G. Piette and B. J. Schroers, Phys. Rev. D 53 (1996) 844; J. Gladikowski, Z. Phys. C 73 (1996) 181.

[5] P. K. Ghosh and S. K. Ghosh, Phys. Lett. B 366 (1996) 199; K. Kimm, K. Lee and T. Lee, Phys. Rev. D 53 (1996) 4436.

[6] P. K. Ghosh, Phys. Lett. B 384 (1996) 185; K. Kimm, K. Lee and T. Lee, Phys. Lett. B 380 (1996) 303.

[7] K. Arthur, D. H. Tchrakian and Y. Yang, Phys. Rev. D 54 (1996) 5245; Y. Yang, Commun. Math. Phys. 181 (1996) 485; The Existencs of Solitons in Gauged Sigma Models with Broken Symmetry: Some Remarks, To appear.

[8] A. A. Belavin and A. M. Polyakov, JETP Lett. 22 (1975) 245; R. Rajaraman, Solitons and Instantons (North Holland, Amsterdam, 1982); W. J. Zakrzewski, Low Dimensional Sigma Models (Adam Hilger, Bristol, 1989).

[9] G. 't Hooft, Nucl. Phys. B 79 (1974) 276; A. M. Polyakov, JETP Lett. 20 (1974) 94.

[10] J. Hong, Y. Kim and P-Y. Pac, Phys. Rev. Lett. 64 (1990) 2330; R. Jackiw and E. Weinberg, Phys. Rev. Lett. 64 (1990) 2334.

[11] E. R. C. Abraham and P. K. Townsend, Nucl. Phys. B 351 (1991) 313; Phys. Lett. B

291 (1992) 85; Phys. Lett. B 295 (1992) 225; A. Opfermann and G. Papadopoulos, Nucl. Phys. B 488 (1997) 177.

[12] R. A. Leese, Nucl. Phys. B 344 (1990) 33; Nucl. Phys. B 366 (1991) 283.

[13] M. K. Prasad and C. M. Sommerfield, Phys. Rev. Lett. 35 (1975) 760; S. Coleman, S. Parke, A. Neveu and C. M. Sommerfield, Phys. Rev. D 15 (1977) 544.

[14] E. B. Bogomol'nyi, Yad. Fiz. 24 (1976) 861 [Sov. J. Nucl. Phys. 24 (1976) 449].

[15] H- C. Kao, K. Lee and T. Lee, Phys. Rev. D 55 (1997) 6447.